# Dissipative Dynamics of a Josephson Junction In the Bose-Gases

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## Abstract

The dissipative dynamics of a Josephson junction in the Bose-gases is considered within the framework of the model of a tunneling Hamiltonian. The effective action which describes the dynamics of the phase difference across the junction is derived using functional integration method. The dynamic equation obtained for the phase difference across the junction is analyzed for the finite temperatures in the low frequency limit involving the radiation terms. The asymmetric case of the Bose-gases with the different order parameters is calculated as well.

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## I. INTRODUCTION

The experimental realization of Bose-Einstein condensation in atomic vapors [1, 2, 3, 4] has allowed to observe a great variety of macroscopic quantum effects. In particular, there arises a considerable interest to the study of the Josephson effect in the Bose-condensed gases as one of intriguing possibilities to explore the macroscopic quantum effects related directly to the broken symmetry in the quantum systems. The dynamics of the Josephson effect is governed by the difference between the phases of the condensates, playing a role of macroscopic quantum variable.

The theoretical treatment of the Josephson effect includes both the internal effect for atoms of a gas in the different hyperfine states and the case of the Bose-condensates spatially separated with a potential barrier which acts as a tunneling junction. The latter case due to its direct analogy with superconductors seems us more attractive. A lot of work has already been done in this direction. In [5] the behavior of the condensate density near the potential boundary has been discussed and the quasiclassical expression for the current through a potential barrier has been obtained. The articles [6, 7] are devoted to an applicability of the two-mode approximation in the Josephson junction dynamics. Milburn *et al* in [8] have shown an existence of the self-trapping effect as well as the collapse and revival sequence in the relative population. In [9, 10, 11] the nonlinear Josephson dynamics and macroscopic fluctuations have been considered, resulting in the optimum conditions [12] to observe the Josephson oscillations. Zapata *et al* [13] have presented a semiclassical description of the Josephson junction dynamics. The time-dependent variational analysis of the Josephson effect is given in [14].

One of the most interesting and important aspects in the Josephson junction dynamics from both the theoretical and the experimental viewpoints is the dephasing of the Josephson oscillations due to coupling between the macroscopic relative phase variable and the infinite number of the microscopic degrees of freedom [16, 17]. Historically, in the case of the superconducting systems such description of the phase dynamics was developed in the middle of 1980's [18, 19, 20]. The most important result was a successive derivation of the effective action for the relative phase, revealing the key role of the microscopic degrees of freedom in the irreversible dynamics of the superconducting Josephson junctions. From the mathematical point of view the response functions in the effective action, which prove to be

nonlocal in time, give the full information on the dynamics of a junction. The employment of the low frequency expansion for the response functions allows one to obtain the dissipative dynamics of a superconducting junction, involving Josephson energy, renormalization of the junction capacity (inverse effective mass), and resistance (effective friction) of a junction.

For the system of two Bose-condensates connected with a weakly coupled junction, it is very desirable to trace and explore the dynamics of the relative phase, generalizing the method of the derivation of the effective action from the superconducting case to the case of the Bose-condensed systems. As we will show in the next sections, the gapless sound-like spectrum of low energy excitations in the Bose-condensed gases results in a qualitative change of the irreversible phase dynamics compared with that of the superconducting junctions. So, the main aim of the paper is to derive the effective action for the Bose point-like junction within the framework of the functional integration method in order to find the explicit expressions for the response functions and analyze the low frequency dynamics of a Bose junction.

The plan of the article is the following. First, we derive the general expression for the effective action depending only on the relative phase for the system of two Bose-condensates connected by a point-like junction. Then we consider the case of zero temperature. As a next step, we investigate the effect of finite temperature on the phase dynamics. In addition, from the low frequency expansion of the response functions we find the Josephson energy, renormalization of the effective mass, friction coefficient, and the radiation corrections. The latter can be interpreted as a sound emission from the region of a Bose junction. Finally, we present the case of an asymmetric junction in the Appendix and summarize the results in the Conclusion.

## II. EFFECTIVE ACTION

First, it may be useful to make some remarks on the geometry of the Bose junction and condensates. We keep in mind the case of a point or weakly coupled junction due to a large potential barrier between the two macroscopic infinite reservoirs containing Bose-condensates. So, we can neglect the feed-back effect of the junction on the Bose-condensates and assume that the both condensates are always in the thermal equilibrium state with the constant density depending on the temperature alone. The traditional image of such system

is two bulks with one common point through which the transmission of particles is only possible with some tunneling amplitude.

So, our starting point is the so-called tunneling Hamiltonian ($\hbar = 1$, volume $V = 1$)

$$H = H_l + H_r + H_u + H_t, \tag{1}$$

where $H_{l,r}$ describes the bulk Bose-gas on the left-hand and right-hand sides, respectively,

$$H_{l,r} = \int d^3r \ \Psi^+_{l,r} \left( -\frac{\Delta}{2m} - \mu + \frac{u_{l,r}}{2} \Psi^+_{l,r} \Psi_{l,r} \right) \Psi_{l,r}. \tag{2}$$

The coupling constant $u_{l,r} = 4\pi a_{l,r}/m$ where as usual $a_{l,r}$ is the scattering length. The energy

$$H_u = \frac{U}{2} \left( \frac{N_l - N_r}{2} \right)^2, \tag{3}$$

is analogous to the capacity energy of a junction in the case of superconductors. The constant $U$ can be associated with the second derivative of the total energy $E = E(N_l, N_r)$ with respect to the relative change in the number of particles across the junction

$$U = \left( \frac{\partial^2}{\partial N_l^2} + \frac{\partial^2}{\partial N_r^2} \right) E, \tag{4}$$

and usually is estimated as $U = (\partial \mu_l / \partial N_l + \partial \mu_r / \partial N_r)$ [13]. In general, it may depend on the concrete type of the Bose-junction and simply describes that the energy of the system on the whole may depend on the relative number of particles from each bulk. The total number of the particles in each bulk is given by

$$N_{l,r} = \int d^3r \ \Psi^+_{l,r} \Psi_{l,r}. \tag{5}$$

The term

$$H_t = - \int\limits_{r \in l, r' \in r} d^3r d^3r' \ \left[ \Psi^+_l (\mathbf{r}) \, I (\mathbf{r}, \mathbf{r}') \, \Psi_r (\mathbf{r}') + h.c. \right] \tag{6}$$

is responsible for the transitions of particles from the right-hand to the left-hand bulk and *vice versa*.

To study the properties of the system described by (1), we calculate the partition function using the analogy of the superconducting junction

$$Z = \int \mathcal{D}^2 \Psi_l \mathcal{D}^2 \Psi_r \ \exp \left[ -S_E \right], \tag{7}$$

where the action on the Matsubara (imaginary) time reads

$$
\begin{gathered}
S_E = \int\limits_{-\beta/2}^{\beta/2} d\tau \; L_E, \\
L_E = \int d^3r \left\{ \Psi_l^+ \, \tfrac{\partial}{\partial \tau} \; \Psi_l + \Psi_r^+ \, \tfrac{\partial}{\partial \tau} \; \Psi_r \right\} + H.
\end{gathered}
\tag{8}
$$

To eliminate the quartic term in the action which comes from the "capacity" energy $H_u$, we use the Hubbard-Stratonovich procedure by introducing an additional gauge field $V(\tau)$ on the analogy with the so-called plasmon gauge field in metals

$$
\begin{gathered}
\exp\left[ -\tfrac{U}{2} \int d\tau \; \left( \tfrac{N_l - N_r}{2} \right)^2 \right] = \int \mathcal{D}V \exp\left\{ -\int d\tau \; \left[ \tfrac{V^2(\tau)}{2U} + i \tfrac{N_l - N_r}{2} V\left(\tau\right) \right] \right\}, \\
\int \mathcal{D}V \exp\left[ -\int d\tau \; \tfrac{V^2(\tau)}{2U} \right] = 1.
\end{gathered}
\tag{9}
$$

Next, we follow the Bogoliubov method of separating the field operators into the condensate and non-condensate fractions, i.e., $\Psi_{l,r} = c_{l,r} + \Phi_{l,r}$. Denoting $x = (\tau, \mathbf{r})$ and introducing convenient Nambu spinor notations for the field operators and, correspondingly, matrices for the Green functions and tunneling amplitudes, we arrive at the following expression for the partition function

$$
Z = \int \mathcal{D}V \mathcal{D}^2 C \exp\left[ -S_0 \right] \int \mathcal{D}^2 \Phi \exp\left[ -S_\Phi \right]. \tag{10}
$$

Thus, we split the initial action into the two parts which correspond to the condensate and noncondensate fields, respectively,

$$
\begin{gathered}
S_0 = \int d\tau \begin{pmatrix} c_l^+ \left( \frac{\partial}{\partial \tau} - \mu + i \frac{V(\tau)}{2} \right) c_l + \frac{u_{l,r}}{2} c_l^+ c_l^+ c_l c_l + \left( l \to r, V \to -V \right) \\ -I_0 \left( c_l^+ c_r + c_r^+ c_l \right) + \frac{V^2(\tau)}{2U} \end{pmatrix}, \\
\\
S_\Phi = \int dx dx' \left\{ \Phi^+ \left( \widetilde{G}^{(0)-1} - \widetilde{I} \right) \Phi - C^+ \widetilde{I} \Phi - \Phi^+ \widetilde{I} C \right\}.
\end{gathered}
\tag{11}
$$

For the field operators we used the spinor notations

$$
\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Xi_l \\ \Xi_r \end{pmatrix}, \; \Xi_{l,r} = \begin{pmatrix} \Phi_{l,r} \\ \Phi_{l,r}^+ \end{pmatrix}, \; C = \frac{1}{\sqrt{2}} \begin{pmatrix} C_l \\ C_r \end{pmatrix}, \; C_{l,r} = \begin{pmatrix} c_{l,r} \\ c_{l,r}^+ \end{pmatrix}. \tag{12}
$$

In the expression for the condensate part $S_0$ we define the amplitude $I_0$ equal to

$$
I_0 = \int d^3r \, d^3r' \, I(\mathbf{r}, \mathbf{r}'). \tag{13}
$$

and corresponding to the tunneling process of the condensate-to-condensate particles. It is straightforward to obtain the following expressions for the matrix Green functions

$$
\widetilde{G}^{(0)-1} = \begin{pmatrix} \widehat{G}_l^{(0)-1} & 0 \\ 0 & \widehat{G}_r^{(0)-1} \end{pmatrix} \delta\left(x - x'\right), \tag{14}
$$

$$\widehat{G}_{l,r}^{(0)-1} = \begin{pmatrix} G_{l,r}^{(0)-1} + \Sigma_{11}^{l,r} & \Sigma_{20}^{l,r} \\ \Sigma_{02}^{l,r} & \overline{G}_{l,r}^{(0)-1} + \overline{\Sigma}_{11}^{l,r} \end{pmatrix}, \tag{15}$$

where the inverse Green functions and self-energy parts are given by the well-known expressions

$$\begin{gathered} G_{l,r}^{(0)-1} = \frac{\partial}{\partial \tau} - \frac{\Delta}{2m} - \mu \pm i\frac{V(\tau)}{2}, \\ \Sigma_{11}^{l,r} = 2u_{l,r}\, c_{l,r}^{+} c_{l,r},\ \Sigma_{20}^{l,r} = u_{l,r}\, c_{l,r} c_{l,r},\ \Sigma_{02}^{l,r} = u_{l,r}\, c_{l,r}^{+} c_{l,r}^{+}. \end{gathered} \tag{16}$$

Accordingly, the matrix Green function $\widehat{G}_{l,r}^{(0)}$ can be represented as

$$\widehat{G}_{l,r}^{(0)} = \begin{pmatrix} G_{l,r} & F_{l,r} \\ F_{l,r}^{+} & \overline{G}_{l,r} \end{pmatrix}. \tag{17}$$

The transfer matrix here has the form

$$\widetilde{I} = \begin{pmatrix} 0 & \widehat{I} \\ \widehat{I}^{*} & 0 \end{pmatrix},\ \ \widehat{I} = \begin{pmatrix} I & 0 \\ 0 & \overline{I} \end{pmatrix},\ \ I = I\left(x, x'\right) = I\left(\mathbf{r}, \mathbf{r}'\right)\delta\left(\tau - \tau'\right). \tag{18}$$

As one can readily see, if we employ a gauge transformation of the field operators

$$\Psi_{l,r} \to \exp\left[i\varphi_{l,r}\left(\tau\right)\right]\Psi_{l,r}, \tag{19}$$

and impose the conditions $\dot{\varphi}_l = -V/2$, $\dot{\varphi}_r = V/2$, i.e.,

$$\dot{\varphi} = V,\ \ \varphi = \varphi_r - \varphi_l, \tag{20}$$

both normal $G_{l,r}$ and anomalous Green functions $F_{l,r}$ (17) gain additional phase factors with respect to the functions in the lack of an external field (notations of [21]).

$$\begin{gathered} G_{l,r}\left(\tau, \tau'\right) \to \exp\left(i[\varphi_{l,r}\left(\tau\right) - \varphi_{l,r}\left(\tau'\right)]\right) G_{l,r}\left(\tau - \tau'\right), \\ F_{l,r}\left(\tau, \tau'\right) \to \exp\left(i[\varphi_{l,r}\left(\tau\right) + \varphi_{l,r}\left(\tau'\right)]\right) F_{l,r}\left(\tau - \tau'\right). \end{gathered} \tag{21}$$

The part of action $S_\Phi$ in (11) is quadratic in the non-condensate field operators so we can integrate them out. To perform the integration, we employ the well-known formula

$$\int \mathcal{D}^2\Phi \exp\left[-\Phi^{+}\alpha\Phi + \beta^{+}\Phi + \Phi^{+}\beta\right] = \exp\left[\beta^{+}\alpha^{-1}\beta - \mathrm{Tr}\left[\ln\left(\alpha\right)\right]\right], \tag{22}$$

with $\alpha = G^{(0)-1} - \widetilde{I}$ and $\beta = \widetilde{I}C$ to arrive at the partition function

$$Z = \int \mathcal{D}\varphi \mathcal{D}^2 C \exp\left[-S\right] \tag{23}$$

with the effective action given by

$$S = S_0 - \mathrm{Tr}\left[C^+ \widetilde{I}\left(\widetilde{G}^{(0)-1} - \widetilde{I}\right)^{-1} \widetilde{I}C\right] + \mathrm{Tr}\left[\ln\left(\widetilde{G}^{(0)-1} - \widetilde{I}\right)\right]. \tag{24}$$

In order to run analytically further, it is necessary to make the following approximations. First, we expand the second and third terms of (24) in powers of the tunneling amplitude $I$ to the first nonvanishing order. Then, as is stated above, we consider the simplest case of a point-like junction putting $I\left(x, x'\right) = I_0 \delta\left(\mathbf{r}\right) \delta\left(\mathbf{r}'\right) \delta\left(\tau - \tau'\right)$. The latter also allows us to escape from the problem of summing all higher-order terms in the tunneling amplitude $I$, which is inherent in a junction of the plane geometry [17, 22] with the conservation of the tangential components of the momentum of a tunneling particle. The problem in essence becomes one-dimensional [15, 22] and results in a strongly dissipative low-frequency dynamics independent of the tunneling amplitude and governed by the bulk relaxation alone. In our consideration this would correspond to the amplitude independent on the $x$ and $y$ coordinates, i.e., $I \propto \delta(z)\delta(z')$. Finally, the third approximation we use is a saddle-point approximation for the condensate part of the partition function. Substituting $c_{l,r} = \sqrt{n_{0l,r}}$ where $n_{0l,r}$ is the density of particles in the condensate fraction, we obtain the expression for the partition function depending on the phase difference alone

$$Z_\varphi = \int \mathcal{D}\varphi \;\; \exp\left(-S_{eff}\left[\varphi\right]\right). \tag{25}$$

The corresponding effective action reads

$$\begin{array}{l} S_{eff}\left[\varphi\right] = \int d\tau \;\; \left[\frac{1}{2U}\left(\frac{d\varphi}{d\tau}\right)^2 - 2I_0\sqrt{n_{0l}n_{0r}}\cos\varphi\right] - \\ I_0^2 \int d\tau d\tau' \;\; \left\{\begin{array}{l} \alpha\left(\tau - \tau'\right)\cos\left[\varphi\left(\tau\right) - \varphi\left(\tau'\right)\right] \\ +\beta\left(\tau - \tau'\right)\cos\left[\varphi\left(\tau\right) + \varphi\left(\tau'\right)\right] \end{array}\right\}. \end{array} \tag{26}$$

Here the response functions can be written using the Green functions

$$\begin{array}{c} \alpha\left(\tau\right) = n_{0r} g_l^+\left(\tau\right) + n_{0l} g_r^+\left(\tau\right) + G\left(\tau\right), \\ \\ \beta\left(\tau\right) = n_{0r} f_l\left(\tau\right) + n_{0l} f_r\left(\tau\right) + F\left(\tau\right), \end{array} \tag{27}$$

where

$$g^{\pm}_{l,r}(\tau) = \int \frac{d^3p}{2(2\pi)^3}\left[G_{l,r}(p,\tau) \pm G_{l,r}(p,-\tau)\right],\; f_{l,r}(\tau) = \int \frac{d^3p}{(2\pi)^3} F_{l,r}(p,\tau)\,,$$

$$G(\tau) = 2\left[g^{+}_{l}(\tau)\, g^{+}_{r}(\tau) - g^{-}_{l}(\tau)\, g^{-}_{r}(\tau)\right],\; F(\tau) = 2 f_l(\tau)\, f_r(\tau)\,. \tag{28}$$

The Fourier components of the Green functions of a weakly interacting Bose-gas are given by [21]

$$G_{l,r}(p,\omega_n) = \frac{i\omega_n+\xi_p+\Delta_{l,r}}{\omega_n^2+\varepsilon^2_{l,r}(p)},\; F_{l,r}(p,\omega_n) = -\frac{\Delta_{l,r}}{\omega_n^2+\varepsilon^2_{l,r}(p)},$$

$$\varepsilon^2_{l,r}(p) = \xi_p^2 + 2\Delta_{l,r}\xi_p, \quad \xi_p = \frac{p^2}{2m}\,, \quad \Delta_{l,r} = u_{l,r} n_{0l,r}. \tag{29}$$

Thus, in order to comprehend the dynamics of the relative phase difference $\varphi$ across the junction, one should analyze the behavior of the response functions $\alpha$ and $\beta$ as a function of time.

## III. THE RESPONSE FUNCTIONS.

The calculation of the response functions in the general form is a rather complicated problem. However, keeping in view, first of all, the study of the low frequency dynamics of a junction, we can restrict our calculations by analyzing the behavior of the response functions on the long-time scale. This means that we should find the low frequency decomposition of the response functions in the Matsubara frequencies. Next, we will use the procedure of analytical continuation in order to derive the dynamic equation which the relative phase $\varphi$ obeys.

### A. Fourier transformation of the $\alpha$-response.

From the analysis of the Green function behavior $g^{\pm}_{l,r}(\tau)$ it follows that zero Fourier component of the $\alpha$-response function diverges. The simplest way to avoid this obstacle is to deal with the difference $\tilde{\alpha}(\omega_n) = \alpha(\omega_n) - \alpha(0)$. This corresponds to the substitution $\alpha(\tau) = \tilde{\alpha}(\tau) + \alpha(0)\delta(\tau)$ into effective action (26) and the second term $\alpha(0)\delta(\tau)$ yields a physically unimportant time-independent contribution into the action, meaning a shift of the ground state energy of a junction. Following this procedure, we can arrive at the explicit expressions of the Fourier components for all the terms in (27).

For the first two terms in the $\alpha$-response function of (27), we have a simple formula

$$\tilde{g}^{+}(\omega_n) = -\frac{\pi\nu}{\sqrt{2}}\left[\sqrt{1+\left|\frac{\omega_n}{\Delta}\right|}-1\right], \tag{30}$$

and the corresponding expansion to third order in $\omega/\Delta$ is

$$\tilde{g}^{+}(\omega_n \to 0) \approx -\frac{\pi\nu}{2\sqrt{2}}\left|\frac{\omega_n}{\Delta}\right|\left[1-\frac{1}{4}\left|\frac{\omega_n}{\Delta}\right|+\frac{1}{8}\left|\frac{\omega_n}{\Delta}\right|^2-\ldots\right]. \tag{31}$$

Here, $\nu = m\sqrt{m\Delta}/(\sqrt{2}\pi^2)$ is the density of states at the energy equal to $\Delta$ in the normal gas.

In the case of $\tilde{G}$ the calculations of the Fourier components are much more complicated. So, we could find the expressions only in the case of zero temperature and the first nonvanishing order in temperature. Note that nonzero temperature effects are connected with the behavior of $\tilde{G}(\omega) = \tilde{G}_0(\omega) + \tilde{G}_T(\omega)$.

For zero-temperature part of $\tilde{G}(\omega)$, we obtain

$$\begin{array}{l}\tilde{G}_0(\omega) = -\nu_l\nu_r\Delta_l\Delta_r\omega^2 \int\limits_0^\infty\int\limits_0^\infty \frac{dxdy\,\sqrt{(\sqrt{x^2+1}-1)(\sqrt{y^2+1}-1)}}{\left[(\Delta_l x+\Delta_r y)^2+\omega^2\right](\Delta_l x+\Delta_r y)} \\ \left(1-\frac{xy}{\sqrt{(x^2+1)(y^2+1)}}\right).\end{array} \tag{32}$$

Unfortunately, we could not evaluate expression (32) in the explicit analytic form. Thus we report here the Fourier expansion up to third order in $\omega$

$$\tilde{G}_0(\omega \to 0) = -\frac{\pi\nu_l\nu_r\omega^2}{8\sqrt{\Delta_l\Delta_r}}\left[\phi\left(\frac{\Delta_l-\Delta_r}{\Delta_l+\Delta_r}\right)-\frac{1}{3}\frac{|\omega|}{\sqrt{\Delta_l\Delta_r}}+\ldots\right]. \tag{33}$$

Here we introduced the function

$$\begin{array}{l}\phi(q) = \frac{2\pi}{3}+\frac{1}{q^2}\left(1-3q^2-\sqrt{1-q^2}\right)-\frac{2}{\pi}\ln^2\left[\sqrt{\frac{1+q}{1-q}}\right]- \\ \quad\frac{2}{\pi}\left(\begin{array}{l}\mathrm{Li}_2\left[-\sqrt{\frac{1+q}{1-q}}\right]+\mathrm{Li}_2\left[-\sqrt{\frac{1-q}{1+q}}\right]+ \\ \mathrm{Li}_2\left[1-\sqrt{\frac{1+q}{1-q}}\right]+\mathrm{Li}_2\left[1-\sqrt{\frac{1-q}{1+q}}\right]\end{array}\right),\end{array} \tag{34}$$

where $\mathrm{Li}_2(\mathrm{z}) = \int_z^0 dt\ln(1-t)/t$ is the dilogarithm function. The Taylor expansion of $\phi(q)$ in the case of $|q| \ll 1$ reads:

$$\phi(q) \approx \pi - \frac{5}{2}+\frac{q^2}{8}+\ldots \tag{35}$$

In the case of the equal order parameters $\Delta = \Delta_l = \Delta_r$ we obtain:

$$\tilde{G}_0(\omega \to 0) \approx -\frac{\pi\nu^2\Delta}{8}\left|\frac{\omega}{\Delta}\right|^2\left[\pi-\frac{5}{2}-\frac{1}{3}\left|\frac{\omega}{\Delta}\right|+\ldots\right]. \tag{36}$$

The other finite-temperature contribution into $\tilde{G}$ can readily be evaluated as

$$\tilde{G}_T(\omega) = \frac{\pi\nu_l\nu_r\sqrt{\Delta_l\Delta_r}}{24}\left(\frac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\left[\sqrt{\frac{\Delta_l}{\Delta_r}}+\sqrt{\frac{\Delta_r}{\Delta_l}}-\sqrt{\frac{\Delta_l+|\omega_n|}{\Delta_r}}-\sqrt{\frac{\Delta_r+|\omega_n|}{\Delta_l}}\right]+\ldots, \tag{37}$$

which in the case of $\Delta = \Delta_l = \Delta_r$ gives:

$$\tilde{G}_T(\omega) = \frac{\pi\nu^2\Delta}{12}\left(\frac{2\pi T}{\Delta}\right)^2\left[1-\sqrt{1+\left|\frac{\omega_n}{\Delta}\right|}\right]+\ldots. \tag{38}$$

Expanding this expression to third order in $\omega_n/\Delta$ and including zero temperature terms yields

$$\tilde{G}(\omega_n\to 0) \approx -\frac{\pi\nu^2\Delta}{24}\left|\frac{\omega_n}{\Delta}\right|\left[\begin{array}{l}\left(\frac{2\pi T}{\Delta}\right)^2+\left|\frac{\omega_n}{\Delta}\right|\left[3\pi-\frac{15}{2}-\frac{1}{4}\left(\frac{2\pi T}{\Delta}\right)^2\right]- \\ \left|\frac{\omega_n}{\Delta}\right|^2\left[1-\frac{1}{8}\left(\frac{2\pi T}{\Delta}\right)^2\right]-\ldots\end{array}\right]. \tag{39}$$

It is worthwhile to emphasize the appearance a linear term in $|\omega_n|$ in Eq.(39) at finite temperatures. As we will see below, this results in the manifestation of an additional temperature-dependent contribution into the dissipation of the junction. The effect can be interpreted as a tunneling of normal thermal excitations existing in the system due to finite temperatures.

### B. Fourier transformation of the $\beta$-response.

Like the preceding section, we can evaluate the Fourier components and find the low frequency expansion for the anomalous Green functions $f(\tau)$. For the first two terms in the $\beta$-response function (27), we obtain

$$f(\omega_n) = -\frac{\pi\nu}{\sqrt{2}\sqrt{1+|\omega_n/\Delta|}} \underset{\omega_n\to 0}{\approx} -\frac{\pi\nu}{\sqrt{2}}\left(1-\frac{1}{2}\left|\frac{\omega_n}{\Delta}\right|+\frac{3}{8}\left|\frac{\omega_n}{\Delta}\right|^2-\frac{5}{16}\left|\frac{\omega_n}{\Delta}\right|^3+\ldots\right). \tag{40}$$

The calculation of $F(\omega_n)$ requires a special attention. In fact, analyzing the formula of the Fourier transform

$$F(\omega_n) = \frac{\pi^2\nu_l\nu_r\sqrt{\Delta_l\Delta_r}}{\beta}\sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{(\Delta_l+|\omega_k|)(\Delta_r+|\omega_k-\omega_n|)}}, \tag{41}$$

we see that the zero-temperature expression

$$F(\omega) = \frac{\pi\nu_l\nu_r\sqrt{\Delta_l\Delta_r}}{2}\int_{-\infty}^{\infty}d\omega'\frac{1}{\sqrt{(\Delta_l+|\omega'|)(\Delta_r+|\omega'-\omega|)}}, \tag{42}$$

diverges logarithmically at $\omega = 0$ due to behavior of the integrand at large frequencies $\omega' \to \infty$. To perform integration over $\omega'$, we should first pay attention that so far we could neglect the dependence on the momentum in the self-energy parts since all the integrals gain their values in the region of small momenta. In fact, the self-energy parts depend on the momentum and we must use the exact expressions $\Sigma_{20}(p) = \Sigma_{02}(p) = n_0 U(p)$ where $U(p)$ is a Fourier component of the interaction between particles, $\Sigma_{20}(0)$ being $n_0 U(0) = 4\pi a n_0/m$. Here $a$ is the scattering length, $n_0$ is the particle density, and $m$ is the mass of a particle. The order of the magnitude for the momentum at which $U(p)$ decays can be estimated roughly as a reciprocal of the scattering length, i.e., $p \simeq 1/a$. So, we put the upper limit of integration equal to the cutoff frequency $\omega_c \simeq 1/ma^2$ within the logarithmic accuracy.

Finally, representing $F(\omega_n)$ as a sum of zero-temperature and finite temperature contributions $F(\omega_n) = F_0(\omega_n) + F_T(\omega_n)$, we find the zero-temperature term

$$F_0(\omega_n) = \frac{\pi\nu_l\nu_r\sqrt{\Delta_l\Delta_r}}{2}\left(\begin{array}{c} 2\ln\left[\frac{4\omega_c}{\left(\sqrt{\Delta_l}+\sqrt{\Delta_r}\right)^2}\right] - 2\ln\left[\frac{\left(\sqrt{\Delta_l}+\sqrt{\Delta_r+|\omega_n|}\right)\left(\sqrt{\Delta_r}+\sqrt{\Delta_l+|\omega_n|}\right)}{\left(\sqrt{\Delta_l}+\sqrt{\Delta_r}\right)^2}\right] + \\ + \arctan\frac{|\omega_n|+\Delta_l-\Delta_r}{2\sqrt{(|\omega_n|+\Delta_l)\Delta_r}} + \arctan\frac{|\omega_n|+\Delta_r-\Delta_l}{2\sqrt{(|\omega_n|+\Delta_r)\Delta_l}} \end{array}\right). \tag{43}$$

and the finite temperature one, respectively

$$F_T(\omega_n) = \frac{\pi\nu_l\nu_r\sqrt{\Delta_l\Delta_r}}{24}\left(\frac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\left(\begin{array}{c} \frac{\Delta_l}{\sqrt{\Delta_r(\Delta_l+|\omega_n|)}} + \frac{\Delta_r}{\sqrt{\Delta_l(\Delta_r+|\omega_n|)}} \\ -\frac{\Delta_l+\Delta_r}{2\sqrt{\Delta_l\Delta_r}} \end{array}\right) + \ldots. \tag{44}$$

These functions in the case of $\Delta_l = \Delta_r = \Delta$ take the form:

$$F_0(\omega_n) = \pi\nu^2\Delta\left(2\ln\left[\frac{2\sqrt{\omega_c/\Delta}}{1+\sqrt{1+|\omega_n/\Delta|}}\right] + \arctan\frac{|\omega_n/\Delta|}{2\sqrt{1+|\omega_n/\Delta|}}\right), \tag{45}$$

$$F_T(\omega_n) = \frac{\pi\nu^2\Delta}{12}\left(\frac{2\pi T}{\Delta}\right)^2\left(\frac{1}{\sqrt{1+|\omega_n/\Delta|}} - \frac{1}{2}\right) + \ldots. \tag{46}$$

Expanding these expressions to third order in $\omega_n/\Delta$ yields

$$F(\omega_n \to 0) \approx \frac{\pi\nu^2\Delta}{24}\left(\begin{array}{l} 24\ln\left[\frac{\omega_c}{\Delta}\right] + \left(\frac{2\pi T}{\Delta}\right)^2 - \left|\frac{\omega_n}{\Delta}\right|\left(\frac{2\pi T}{\Delta}\right)^2 - \\ \left|\frac{\omega_n}{\Delta}\right|^2\left[\frac{3}{2} - \frac{3}{4}\left(\frac{2\pi T}{\Delta}\right)^2\right] + \\ \left|\frac{\omega_n}{\Delta}\right|^3\left[1 - \frac{5}{8}\left(\frac{2\pi T}{\Delta}\right)^2\right] - \ldots \end{array}\right). \tag{47}$$

Here, as in the case of $\alpha$-response we emphasize the appearance a linear term in $|\omega_n|$ in Eq. (47) at finite temperatures which contributes to the dissipation and interpreted as a tunneling of normal thermal excitations existing in the system at finite temperatures.

### C. Functional series of the response functions.

The Fourier components of the response functions in the form of a series in $|\omega_n|$ up to third order can be written in the form

$$
\begin{aligned}
\tilde{\alpha}\left(\omega_n\right) &= -\alpha_1|\omega_n| + \alpha_2|\omega_n|^2 - \alpha_3|\omega_n|^3 + \ldots, \\
\beta\left(\omega_n\right) &= -\beta_0 + \beta_1|\omega_n| - \beta_2|\omega_n|^2 + \beta_3|\omega_n|^3 + \ldots.
\end{aligned} \tag{48}
$$

Accordingly, the expressions for the response functions in the imaginary time representation read as

$$
\begin{aligned}
\tilde{\alpha}(\tau) &= \alpha_1 \frac{1}{\pi}\left(\frac{\pi T}{\sin(\pi T\tau)}\right)^2 - \alpha_2\delta''(\tau) - \alpha_3\frac{2(\pi T)^4}{\pi}\left[\frac{3}{\sin^4(\pi T\tau)} - \frac{2}{\sin^2(\pi T\tau)}\right] + \ldots, \\
\beta(\tau) &= -\beta_0\delta(\tau) - \beta_1\frac{1}{\pi}\left(\frac{\pi T}{\sin(\pi T\tau)}\right)^2 + \beta_2\delta''(\tau) + \beta_3\frac{2(\pi T)^4}{\pi}\left[\frac{3}{\sin^4(\pi T\tau)} - \frac{2}{\sin^2(\pi T\tau)}\right] + \ldots.
\end{aligned} \tag{49}
$$

For the sake of brevity, we present expressions for $\alpha_i(T)$ and $\beta_i(T)$ in the symmetric case of $\Delta_l = \Delta_r = \Delta$. The general case of $\Delta_l \neq \Delta_r$ will be considered in the Appendix.

$$
\begin{aligned}
\alpha_1 &= \frac{\gamma}{2}\left\{1 + \frac{1}{3}\sqrt{\frac{na^3}{\pi}}\left(\frac{2\pi T}{\Delta}\right)^2\right\}, \\
\alpha_2 &= \frac{\gamma}{8\Delta}\left\{1 - 4\sqrt{\frac{na^3}{\pi}}\left[\pi - \frac{5}{2} - \frac{1}{12}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}, \\
\alpha_3 &= \frac{\gamma}{16\Delta^2}\left\{1 - \frac{8}{3}\sqrt{\frac{na^3}{\pi}}\left[1 - \frac{1}{8}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}, \\
\beta_0 &= \gamma\Delta\left\{1 - 4\sqrt{\frac{na^3}{\pi}}\left[\ln\left(\frac{1}{na^3}\right) + \frac{1}{24}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}, \\
\beta_1 &= \frac{\gamma}{2}\left\{1 - \frac{1}{3}\sqrt{\frac{na^3}{\pi}}\left(\frac{2\pi T}{\Delta}\right)^2\right\}, \\
\beta_2 &= \frac{3\gamma}{8\Delta}\left\{1 + \frac{2}{3}\sqrt{\frac{na^3}{\pi}}\left[1 - \frac{1}{2}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}, \\
\beta_3 &= \frac{5\gamma}{16\Delta^2}\left\{1 + \frac{8}{15}\sqrt{\frac{na^3}{\pi}}\left[1 - \frac{5}{8}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}.
\end{aligned} \tag{50}
$$

Here $T/\Delta \ll 1$, $\gamma = \sqrt{2}\pi\nu n_0/\Delta$, $n$ is the total density of a gas, $n_0$ is the condensate fraction density, and $a$ is the scattering length. Note that the gas parameter $na^3 \ll 1$ naturally enters the equations.

First of all, from the Fourier expansion of $g(\omega_n)$, $f(\omega_n)$, $G(\omega_n)$, and $F(\omega_n)$ we conclude that the dissipation in a point-like junction due to the presence of the linear $|\omega_n|$ term can be associated with the various physical processes. Thus in the case of the $g$ and $f$ contribution the dissipation can be ascribed to the noncondensate-condensate particle tunneling process and exists down to zero temperature [17]. On the other hand, the inspection of the expressions for $G$ and $F$ shows that these terms, producing no contribution into dissipation

at zero temperature, will be responsible for the explicit $T^2$-behavior of the dissipative effects in the junction. The origin of this finite temperature contribution can be attributed to the tunneling of thermal phonon-like excitations across the junction. The meaning of the other dynamical renormalizations and its temperature behavior will be discussed below.

## IV. JOSEPHSON EQUATION.

To consider the dynamic behavior of the relative phase $\varphi$ in the real time, we now follow the standard procedure of analytical continuation. Accordingly, the substitution $|\omega_n| \to -i\omega$ in the Fourier transform of the Euler-Lagrange equation $\delta S_{eff}/\delta\varphi(\tau) = 0$ in imaginary time entails the classical equation of motion for the relative phase with the next inverse Fourier transformation to the real-time representation. In particular, this means that we should replace $|\omega_n|$ with $-i\omega$ in the above expressions for the Fourier transform of the response functions $\alpha$ and $\beta$. The effective action $S_{eff}[\varphi(t)]$ in the real time, which variation $\delta S_{eff}/\delta\varphi(t) = 0$ yields the real-time equation of motion, can be given by the expression

$$\begin{aligned} S_{eff}\left[\varphi\right] = \int d\,t \,\left[\tfrac{1}{2U}\left(\tfrac{d\varphi}{d\,t}\right)^2 + 2I_0\sqrt{n_{0l}n_{0r}}\cos\varphi\right] - \\ I_0^2 \int d\,t d\,t' \left\{ \begin{array}{l} \tilde{\alpha}\left(t-t'\right)\cos\left[\varphi\left(t\right)-\varphi\left(t'\right)\right] \\ +\beta\left(t-t'\right)\cos\left[\varphi\left(t\right)+\varphi\left(t'\right)\right] \end{array} \right\}. \end{aligned} \tag{51}$$

In the limit of the slowly varying phase the response functions in the real-time representation can be represented in the form of a functional series. According to Eqs.(48), we have

$$\begin{aligned} \tilde{\alpha}(t) &= \alpha_1\delta'(t) - \alpha_2\delta''(t) + \alpha_3\delta'''(t) + \ldots, \\ \beta(t) &= \beta_0\delta(t) - \beta_1\delta'(t) + \beta_2\delta''(t) - \beta_3\delta'''(t) + \cdots. \end{aligned} \tag{52}$$

Employing variational principle to the effective action $S_{eff}[\varphi(t)]$ and using decomposition (52), we can derive the Josephson equations valid for the slow variations of the phase provided the typical time of its evolution is longer than $1/\Delta$:

$$\dddot{\varphi}\, G_3(\varphi) + (3/2)\ddot{\varphi}\dot{\varphi}G_3'(\varphi) - \dot{\varphi}^3 G_3(\varphi) + \ddot{\varphi}G_2(\varphi) + (1/2)\dot{\varphi}^2 G_2'(\varphi) + \dot{\varphi}G(\varphi) + U'(\varphi) = 0, \tag{53}$$

and in accordance with Eq.(20)

$$\dot{\varphi} = -\delta\mu(t) = \mu_1 - \mu_2.$$

Here we have retained the time derivatives of $\varphi(t)$ to third order corresponding to radiation corrections.

In the following, for the sake of brevity we will consider the case of $\Delta_l = \Delta_r = \Delta$. The general expressions for the coefficients for $\Delta_l \neq \Delta_r$ will be considered in the Appendix.

The potential energy of a junction is given by the well-known relation

$$U(\varphi) = -E_J \cos\varphi + (1/2)E_{2J}\cos 2\varphi,$$

with the coefficients

$$E_J = 2n_0 I_0,$$
$$E_{2J} = G_0\Delta\left\{1 - 4\sqrt{\frac{na^3}{\pi}}\left[\ln\frac{1}{na^3} + \frac{1}{24}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}, \tag{54}$$

where

$$G_0 = 2\gamma I_0^2 = 16\pi\sqrt{\frac{na^3}{\pi}}(n_0 I_0/\Delta)^2,$$
$$\Delta = n_0 u. \tag{55}$$

In the above relations the condensate density $n_0 = n_0(T)$ depends explicitly on temperature according to the well-known expression for a depletion of the condensate fraction of the weakly interacting Bose gas [23]. An increase of the temperature leads obviously to decreasing the Josephson energy.

The friction coefficient $G(\varphi)$ determining the Ohmic dissipation is given by

$$G(\varphi) = G_0\left[\cos^2\varphi + \frac{1}{3}\sqrt{\frac{na^3}{\pi}}\left(\frac{2\pi T}{\Delta}\right)^2\sin^2\varphi\right]. \tag{56}$$

It is natural that nonzero temperature enhances the energy dissipation of a junction due to appearance of thermal excitations in the Bose-condensed gas.

We may compare $G_0$ of the Bose-gas to the analogous conductance of a normal Fermi gas with the same density of states:

$$G_N = 4\pi I_0^2\nu^2(\Delta),\ G_N = 8\sqrt{\frac{na^3}{\pi}}G_0. \tag{57}$$

The inverse effective mass of a junction is determined by

$$G_2(\varphi) = U^{-1} - \alpha_2 - \beta_2\cos 2\varphi, \tag{58}$$

$$\alpha_2 = \frac{G_0}{8\Delta}\left\{1 - 4\sqrt{\frac{na^3}{\pi}}\left[\pi - \frac{5}{2} - \frac{1}{12}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\},$$
$$\beta_2 = \frac{3G_0}{8\Delta}\left\{1 + \frac{2}{3}\sqrt{\frac{na^3}{\pi}}\left[1 - \frac{1}{2}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}. \tag{59}$$

The renormalization of the effective mass results from the both condensate-noncondensate and noncondensate-noncondensate particle tunneling processes.

The coefficient $G_3(\varphi)$ responsible for the radiation effects reads

$$G_3(\varphi) = \alpha_3 + \beta_3 \cos 2\varphi,$$
$$\alpha_3 = \frac{G_0}{16\Delta^2}\left\{1 - \frac{8}{3}\sqrt{\frac{na^3}{\pi}}\left[1 - \frac{1}{8}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}, \tag{60}$$
$$\beta_3 = \frac{5G_0}{16\Delta^2}\left\{1 + \frac{8}{15}\sqrt{\frac{na^3}{\pi}}\left[1 - \frac{5}{8}\left(\frac{2\pi T}{\Delta}\right)^2\right]\right\}.$$

These effects can be associated with emitting a sound from the region of a junction during the tunneling of particles across the junction. On the other hand, the radiation effects can be treated as a frequency dispersion of the effective friction coefficient.

We will not enter here in details of the conditions which should be imposed on the coefficients of the Josephson equation (53) in order to observe the well-defined Josephson effect since this topic is already discussed much in the literature. In essence, the necessary condition reduces to the requirement of smallness either quantum zero-point or thermal fluctuations for the phase difference across a junction, i.e., mean square value $\langle(\Delta\varphi)^2\rangle \ll 1$. Involving that $\langle(\Delta\varphi)^2\rangle \sim T/E_J$ in the thermal activation region with the crossover at $T_0 \sim E_J/max\{G, \sqrt{G_2 E_J}\}$ to $\langle(\Delta\varphi)^2\rangle \sim \hbar/max\{G, \sqrt{G_2 E_J}\}$ in the quantum fluctuation regime at lower temperatures, it is desirable to have sufficiently large Josephson energy $E_J$ or, correspondingly, not too large potential barriers.

Another interesting aspect of such kind of experiment is an investigation of the effects beyond the mean-field approximation of a very dilute gas $na^3 \ll 1$, in particular, observation of the temperature effects in the dynamics of a junction. As we have seen, the scale of the temperature effects should reach the order of $\sqrt{na^3}$ at $2\pi T \sim \Delta \ll T_c$. This is of much interest since the temperature behavior of the Josephson dynamics is closely related to the properties of elementary excitations in a condensed Bose-gas.

The gas parameter $na^3$ under the conditions typically realized is at most of order of $10^{-4}$, e.g., $n \sim 10^{15}\text{cm}^{-3}$ and $a = 5\text{nm}$ for $^{87}$Rb. In principle, it is possible to approach $na^3 \sim 1$, increasing $a \to \infty$ by appropriate tuning of magnetic field with the Feshbach resonance. However, too large values of scattering length can facilitate rapid three-body recombination. On the other hand, according to recent work [24] such recombination may not be inevitable at approaching $na^3 \sim 10^{-2}$. For such values of $na^3$, the effects beyond the mean-field approximation becomes well noticeable of about 10%.

The second interesting aspect is connected with the analog of the voltage-current characteristic which is an important attribute of a superconducting junction. This kind of experiment implies maintenance of the constant bias $\delta\mu$ for the chemical potentials across a junction, corresponding to constant pressure drop $\delta P = mn\delta\mu$ and time dependence of the relative phase $\varphi(t) = -\delta\mu t + \varphi_0$. One of possibilities is to use the field of Earth's gravity for this purpose.

As a result, we arrive at the following mean value of the particle current across a Bose junction for a sufficiently large period of time as a function of bias $\delta\mu$

$$\langle I \rangle = \frac{\delta\mu}{\hbar}\langle G(\varphi)\rangle \left[1 - \left(\frac{\delta\mu}{\hbar}\right)^2 \frac{\langle G_3(\varphi)\rangle}{\langle G(\varphi)\rangle}\right], \tag{61}$$

where

$$\langle G(\varphi)\rangle = \frac{1}{2}G_0 \left[1 + \frac{1}{3}\sqrt{\frac{na^3}{\pi}}\left(\frac{2\pi T}{\Delta}\right)^2\right],$$

and $\langle G_3(\varphi)\rangle = \alpha_3$. This experiment may give an information on the dissipative properties of a junction and also on the nonlinear effects to be of the order of $(\delta\mu/\Delta)^3$. Note that nonlinear effects do not contain smallness of $(na^3)^{1/2}$.

## V. CONCLUSION

To summarize, in this paper we have used a functional integration approach for the model of a tunneling Hamiltonian in order to analyze the dynamics of a point-like Josephson junction between two weakly non-ideal Bose gases. The effective action and response functions which describe completely the dynamics of a junction are found. Using the low frequency decomposition of the response functions, the quasiclassical Josephson equation which the time evolution of the phase difference $\varphi$ across the junction obeys is obtained to the terms

of third order in time derivatives. The corresponding kinetic coefficients are calculated analytically, involving the finite-temperature corrections. The temperature effect on the kinetic coefficients demonstrates the $T^2$-behavior.

Like a junction of the planar geometry [15, 17, 22], the dynamics of a point-like junction concerned here has an Ohmic dissipative type due to the gapless character of excitations in a Bose-condensed gas. Note only that the scale of the dissipative effect in the latter case is significantly less.

The behavior of the dissipative particle current on the temperature and the difference of the chemical potentials in Eq. (61) can be used to investigate the role of quasiparticle excitations in the dynamics of a Bose junction. Since the dissipation is closely related to the tunneling process of condensate particles and depends on the structure of a junction, the effect of dissipation in the junction dynamics can be reduced provided the structure of a junction prevents condensate particles from tunneling across the junction. We believe this question deserves further study.

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## VII. APPENDIX

Here we present the general case of $\Delta_l \neq \Delta_r$. To derive expressions for the coefficients in the equation for the phase (53), we first find the Taylor expansion of the response functions

$\alpha$ and $\beta$. Finally, for the coefficients in the functional series (48) we arrive at the expressions:

$$
\begin{aligned}
\alpha_1 &= \tfrac{\gamma}{2}\left\{b_{lr}^{(2)} + \tfrac{1}{3}\left(\tfrac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left(\tfrac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\right\},\\
\alpha_2 &= \tfrac{\gamma}{8\sqrt{\Delta_l\Delta_r}}\left\{b_{lr}^{(3)} - 4\left(\tfrac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[\phi_{lr} - \tfrac{\Delta_l+\Delta_r}{24\sqrt{\Delta_l\Delta_r}}\left(\tfrac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\right]\right\},\\
\alpha_3 &= \tfrac{\gamma}{16\Delta_l\Delta_r}\left\{b_{lr}^{(4)} - \tfrac{8}{3}\left(\tfrac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[1 - \tfrac{\Delta_l^2+\Delta_r^2}{16\Delta_l\Delta_r}\left(\tfrac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\right]\right\},\\
\beta_0 &= \sqrt{\Delta_l\Delta_r}\gamma\left\{b_{lr}^{(1)} - 4\left(\tfrac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[\ln\left[\tfrac{4\omega_c}{\left(\sqrt{\Delta_l}+\sqrt{\Delta_r}\right)^2}\right] + \tfrac{\Delta_l+\Delta_r}{48\sqrt{\Delta_l\Delta_r}}\left(\tfrac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\right]\right\},\\
\beta_1 &= \tfrac{\gamma}{2}\left\{b_{lr}^{(2)} - \tfrac{1}{3}\left(\tfrac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left(\tfrac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\right\},\\
\beta_2 &= \tfrac{3\gamma}{8\sqrt{\Delta_l\Delta_r}}\left\{b_{lr}^{(3)} + \tfrac{2}{3}\left(\tfrac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[\tfrac{4\sqrt{\Delta_l\Delta_r}}{\left(\sqrt{\Delta_l}+\sqrt{\Delta_r}\right)^2} - \tfrac{\Delta_l+\Delta_r}{4\sqrt{\Delta_l\Delta_r}}\left(\tfrac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\right]\right\},\\
\beta_3 &= \tfrac{5\gamma}{16\Delta_l\Delta_r}\left\{b_{lr}^{(4)} + \tfrac{8}{15}\left(\tfrac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[1 - \tfrac{5\left(\Delta_l^2+\Delta_r^2\right)}{16\Delta_l\Delta_r}\left(\tfrac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\right]\right\}.
\end{aligned}
\tag{62}
$$

Here we use the following notations

$$
\phi_{lr} = \phi\left(\frac{\Delta_l - \Delta_r}{\Delta_l + \Delta_r}\right),\ \gamma = \pi\sqrt{\frac{2\nu_l\nu_r n_{0l} n_{0r}}{\Delta_l\Delta_r}}, \tag{63}
$$

$$
b_{lr}^{(n)} = \frac{1}{2}\left[\left(\frac{\Delta_r}{\Delta_l}\right)^{n/2}\left(\frac{n_l a_l^3}{n_r a_r^3}\right)^{1/4} + \left(\frac{\Delta_l}{\Delta_r}\right)^{n/2}\left(\frac{n_r a_r^3}{n_l a_l^3}\right)^{1/4}\right].
$$

and $\phi$ is introduced in (34).

Then, for the potential energy of a junction we have

$$
\begin{aligned}
E_J &= 2I_0\sqrt{n_{0l}n_{0r}},\\
E_{2J} &= G_0\sqrt{\Delta_l\Delta_r}\left\{b_{lr}^{(1)} - 4\left(\tfrac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[\ln\left[\tfrac{4\omega_c}{\left(\sqrt{\Delta_l}+\sqrt{\Delta_r}\right)^2}\right] + \tfrac{\Delta_l+\Delta_r}{48\sqrt{\Delta_l\Delta_r}}\left(\tfrac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\right]\right\}.
\end{aligned}
\tag{64}
$$

$G_0$ is given by

$$
G_0 = 2\gamma I_0^2 = 16\pi\left(\frac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left(I_0\sqrt{\frac{n_{0l}n_{0r}}{\Delta_l\Delta_r}}\right)^2, \tag{65}
$$

$$
\Delta_{l,r} = n_{0l,r}u_{l,r}.
$$

In the above relations the condensate densities $n_0 = n_0(T)$ depend on temperature according to the well-known expression for the depletion of the condensate fraction of the weakly interacting Bose gas [23].

The friction coefficient $G(\varphi)$ determining the Ohmic dissipation is represented by

$$
G(\varphi) = G_0\left[b_{lr}^{(2)}\cos^2\varphi + \frac{1}{3}\left(\frac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left(\frac{2\pi T}{\sqrt{\Delta_l\Delta_r}}\right)^2\sin^2\varphi\right].
$$

It is natural that nonzero temperature enhances the energy dissipation of a junction due to appearance of thermal excitations in the Bose-condensed gas.

Comparing $G_0$ for the Bose-gases with the conductance of normal Fermi gases with the same densities of states, we have

$$G_N = 4\pi I_0^2 \nu_l \nu_r, \; G_N = 8\left(\frac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4} G_0. \tag{66}$$

The inverse effective mass of a junction is determined by

$$G_2(\varphi) = U^{-1} - \alpha_2 - \beta_2 \cos 2\varphi, \tag{67}$$

$$\alpha_2 = \frac{G_0}{8\sqrt{\Delta_l \Delta_r}}\left\{ b_{lr}^{(3)} - 4\left(\frac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[\phi_{lr} - \frac{\Delta_l + \Delta_r}{24\sqrt{\Delta_l \Delta_r}}\left(\frac{2\pi T}{\sqrt{\Delta_l \Delta_r}}\right)^2\right]\right\},$$

$$\beta_2 = \frac{3G_0}{8\sqrt{\Delta_l \Delta_r}}\left\{ b_{lr}^{(3)} + \frac{2}{3}\left(\frac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[\frac{4\sqrt{\Delta_l \Delta_r}}{\left(\sqrt{\Delta_l} + \sqrt{\Delta_r}\right)^2} - \frac{\Delta_l + \Delta_r}{4\sqrt{\Delta_l \Delta_r}}\left(\frac{2\pi T}{\sqrt{\Delta_l \Delta_r}}\right)^2\right]\right\}. \tag{68}$$

The renormalization of the effective mass results from the both condensate-noncondensate and noncondensate-noncondensate particle tunneling processes.

The coefficient $G_3(\varphi)$ responsible for the radiation effects reads

$$G_3(\varphi) = \alpha_3 + \beta_3 \cos 2\varphi,$$

$$\alpha_3 = \frac{G_0}{16\Delta_l \Delta_r}\left\{ b_{lr}^{(4)} - \frac{8}{3}\left(\frac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[1 - \frac{\Delta_l^2 + \Delta_r^2}{16\Delta_l \Delta_r}\left(\frac{2\pi T}{\sqrt{\Delta_l \Delta_r}}\right)^2\right]\right\}, \tag{69}$$

$$\beta_3 = \frac{5G_0}{16\Delta_l \Delta_r}\left\{ b_{lr}^{(4)} + \frac{8}{15}\left(\frac{n_l a_l^3 n_r a_r^3}{\pi^2}\right)^{1/4}\left[1 - \frac{5\left(\Delta_l^2 + \Delta_r^2\right)}{16\Delta_l \Delta_r}\left(\frac{2\pi T}{\sqrt{\Delta_l \Delta_r}}\right)^2\right]\right\}.$$

The expression derived for the dynamical coefficients in equation (53) governing the phase difference across a point-like junction allows us to describe the low frequency dynamics in the asymmetric case of Bose gases with the different order parameters.

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